Programming Languages for Hybrid System Modeling: a new Eldorado?

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ENS

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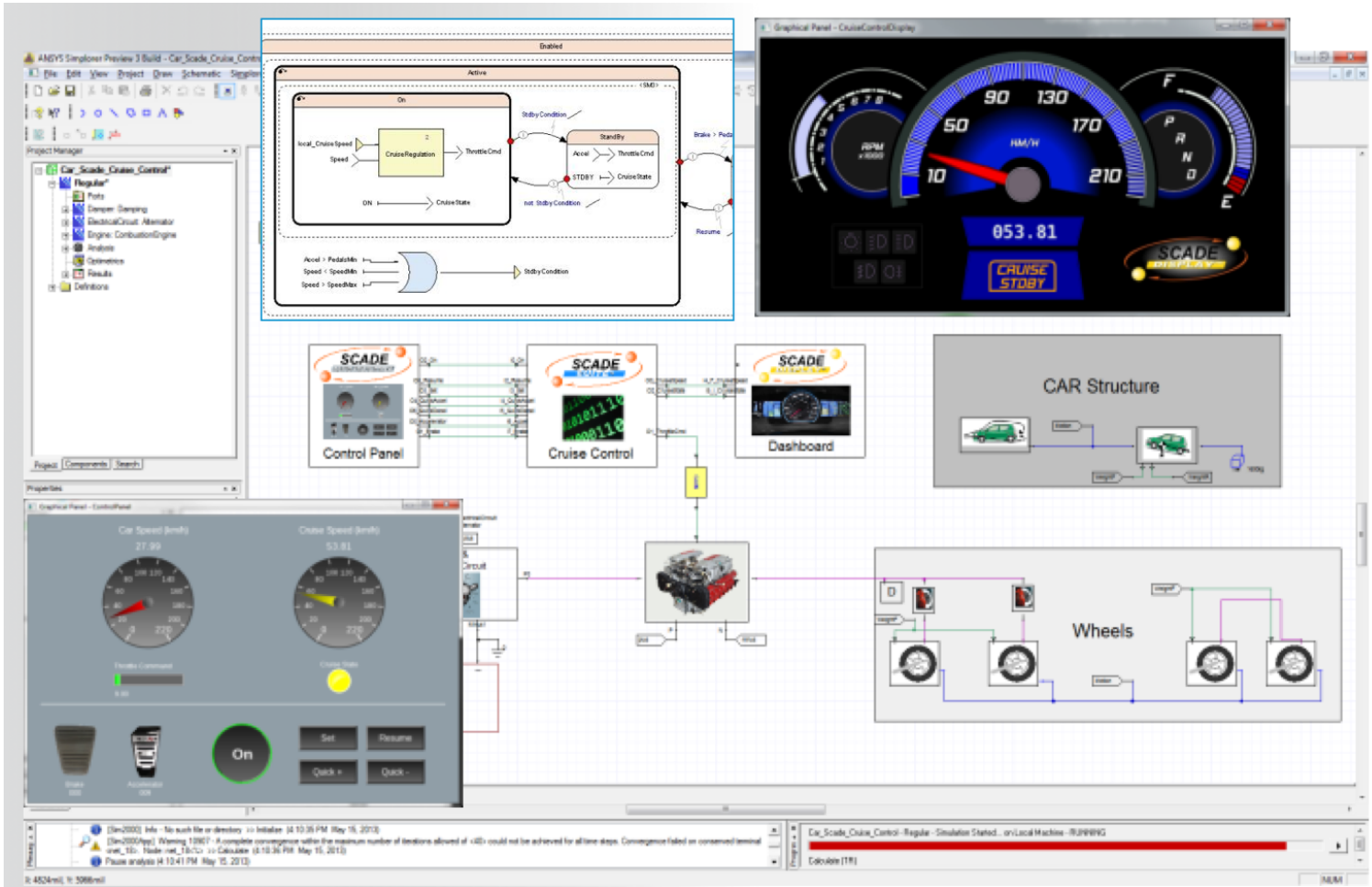
In collaboration with Albert Benveniste, **Timothy Bourke**, Benoît Caillaud and Bruno Pagano.

Hybrid Systems Modeling

Embedded software interacts with physical devices.

The whole system has to be modeled: the controller and the plant.[1]



[1] Image by Esterel-Technologies/ANSYS.

A Wide Range of Hybrid Systems Modelers Exist

Ordinary Differential Equations + discrete time

Simulink/Stateflow ($\geq 10^6$ licences), LabView, Ptolemy II, etc.

Differential Algebraic Equations + discrete time

Modelica, VHDL-AMS, VERILOG-AMS, etc.

Dedicated tools for multi-physics

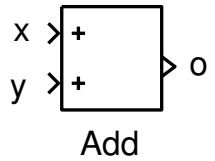
Mechanics, electro-magnetics, fluid, etc.

Co-simulation/combination of tools

- Agree on a common format/protocol: FMI/FMU, S-functions, etc.
- Convert models from one tool to another.

Underlying Mathematical Models

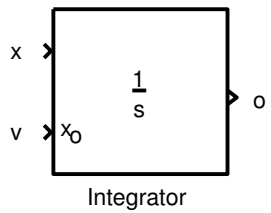
Discrete: synchronous parallelism, sequence equations



- Time is **discrete** and **logical** (indices in $\mathbb{N}$)
- Equation $o = x + y$ means $\forall n \in \mathbb{N}.\ o(n) = x(n) + y(n)$
- $o = \frac{1}{z}(x)$ init v means $o(0) = v$ and $\forall n > 0.\ o(n) = x(n-1)$

(v $\rightarrow$ pre x / v fby x)

Continuous: Ordinary Differential Equations (ODEs)



- Time is **continuous** (indices in $\mathbb{R}$)
- Equation $o = \frac{1}{s}(x)$ init v means $\forall t \in \mathbb{R}.\ o(t) = v(0) + \int_0^t x(\tau)\, d\tau$

Modes: Hierarchical Automata

- Every state can itself contain an automaton and so on.
- Both **discrete-time** and **continuous-time** versions.

Is there anything left to do?

We know how to build tools for discrete-time models.

We know how to build tools for continuous-time models.

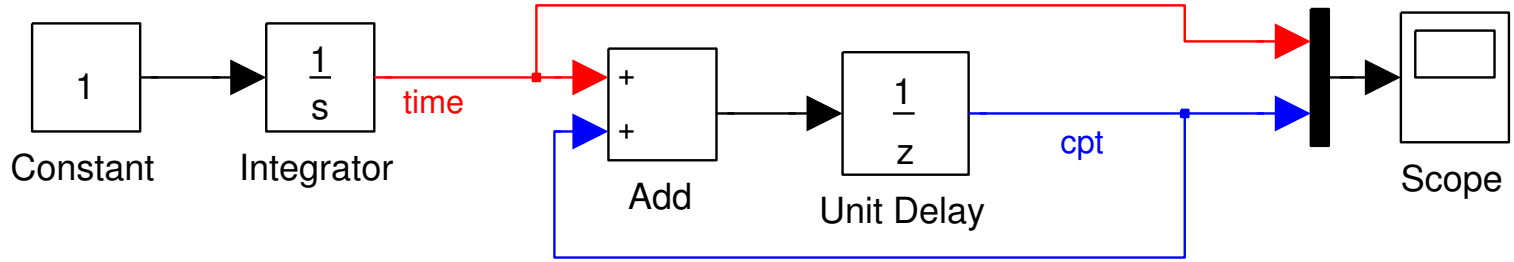
But what if the two are mixed together?

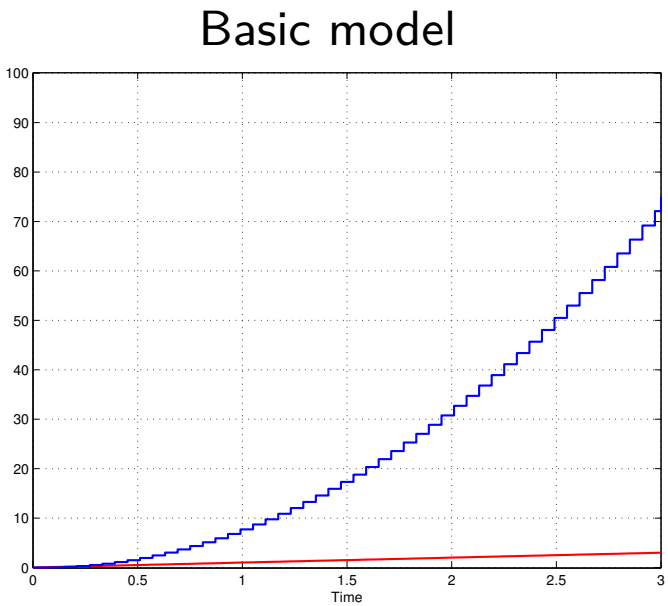
Is it enough to connect one to the other?

Can you trust code automatically generated from such tools?

Tools that mostly work...

Typing issue 1: Mixing continuous & discrete components





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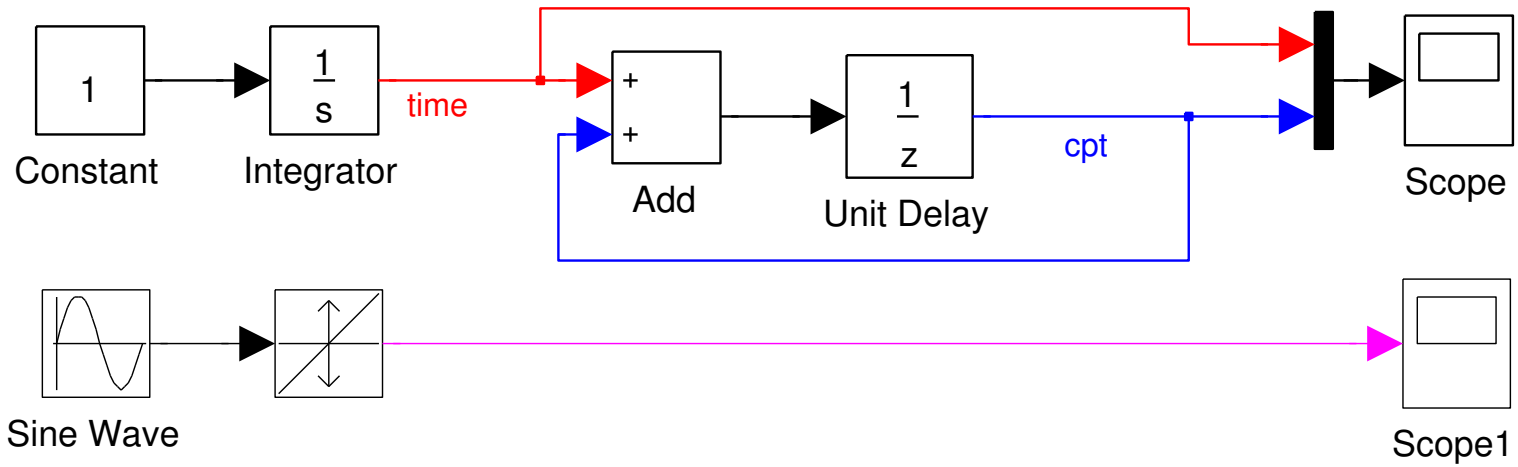
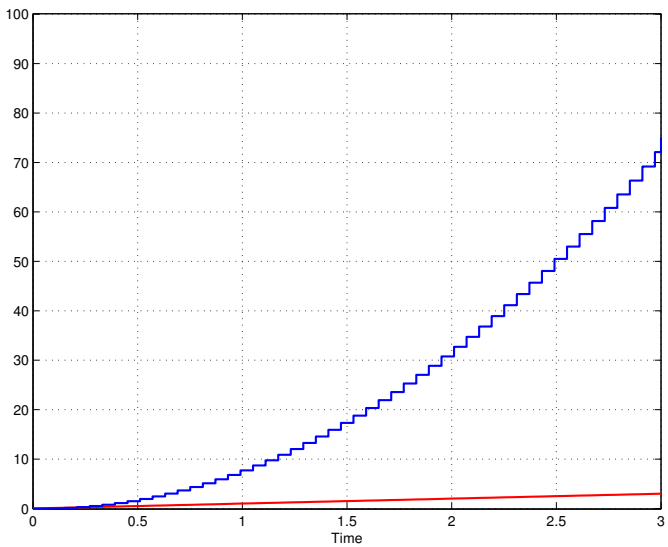
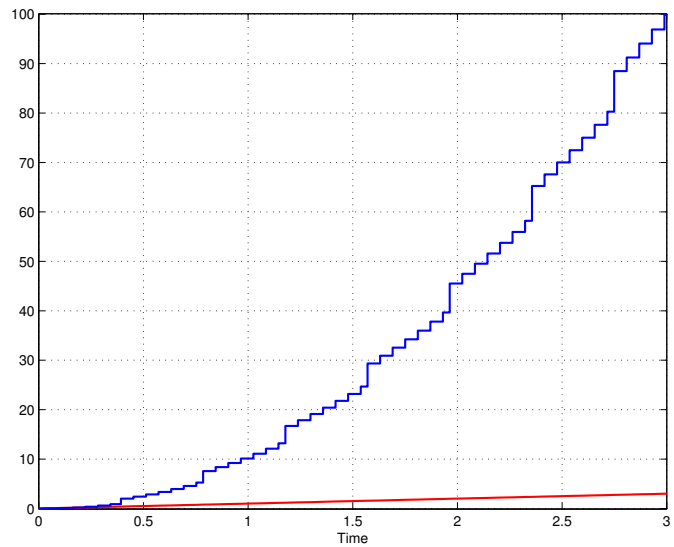


Basic model

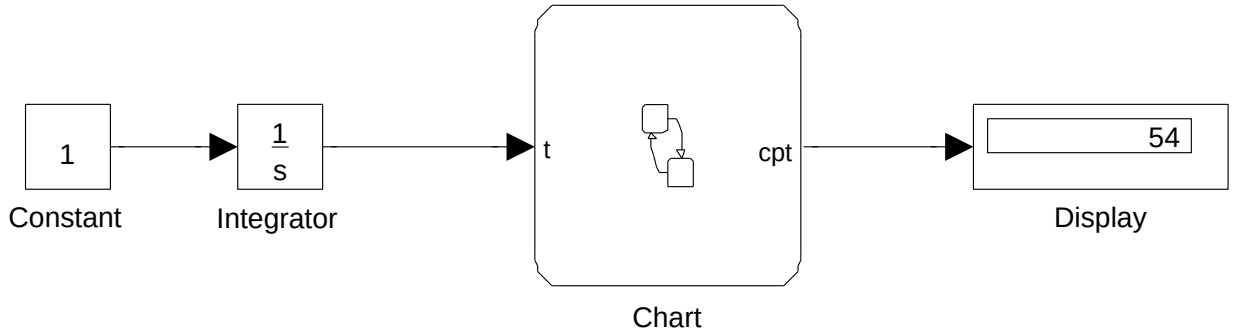


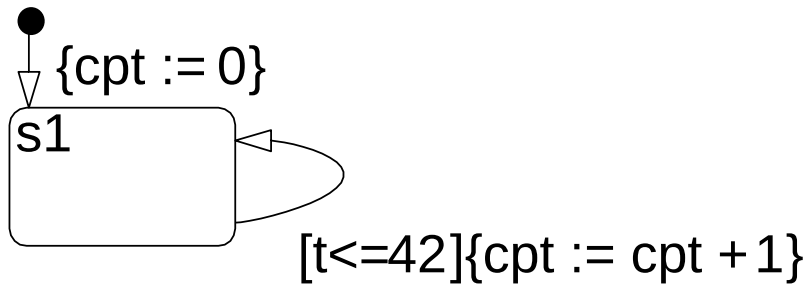
with Sine Wave



- The shape of cpt depends on the steps chosen by the solver.
- Putting another component in parallel can change the result.

Typing issue 2: Boolean guards in continuous automata





How long is a discrete step?

- Adding a parallel component changes the result.
- No warning by the compiler.
- The manual says: “A single transition is taken per major step”.

Here discrete time is not logical
—it comes from the simulation engine.

Continuous automata?

Stateflow User's Guide The Mathworks, pages 16-26 to 16-29, 2011.

Design Considerations for Continuous-Time Modeling in Stateflow Charts

In this section...

Rationale for Design Considerations

To guarantee the integrity — or *smoothness* — of the results in continuous-time modeling, you must constrain your charts to a restricted subset of Stateflow chart semantics. The restricted semantics ensure that inputs do not depend on unpredictable factors — or *side effects* — such as:

- Simulink solver's guess for number of minor intervals in a major time step
- Number of iterations required to stabilize the integration loop or zero crossings loop

By minimizing side effects, a Stateflow chart can maintain its state at minor time steps and, therefore, update state only during major time steps when mode changes occur. Using this heuristic, a Stateflow chart can always compute outputs based on a constant state for continuous-time.

A Stateflow chart generates informative errors to help you correct semantic violations.

Summary of Rules for Continuous-Time Modeling

Here are the rules for modeling continuous-time Stateflow charts:

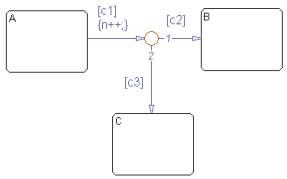
Update local data *only* in transition, `entry`, and `exit` actions

To maintain precision in continuous-time simulation, you should update local data (continuous or discrete) only during physical events at major time steps.

In Stateflow charts, physical events cause state transitions. Therefore, write to local data only in actions that execute during transitions, as follows:

- State `exit` actions, which execute before leaving the state at the beginning of the transition
- Transition actions, which execute during the transition
- State `entry` actions, which execute after entering the new state at the end of the transition
- Condition actions on a transition, but only if the transition directly reaches a state

Consider the following chart.



In this example, the action `{n++}` executes even when conditions `c2` and `c3` are false. In this case, `n` gets updated in a minor time step because there is no state transition.

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Do not call Simulink functions in state `during` actions or transition conditions

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Do not read outputs and derivatives in states or transitions

This restriction ensures smooth outputs in a major time step because it prevents a Stateflow chart from using values that may no longer be valid in the current minor time step. Instead, a Stateflow chart always computes outputs from local discrete data, local continuous data, and chart inputs.

Use discrete variables to govern conditions in `during` actions

This restriction prevents mode changes from occurring between major time steps. When placed in `during` actions, conditions that affect control flow should be governed by discrete variables because they do not change between major time steps.

Do not use input events in continuous-time charts

The presence of input events makes a chart behave like a triggered subsystem and therefore unable to simulate in continuous-time. For example, the following model generates an error if the chart uses a continuous update method.

- 'Restricted subset of Stateflow chart semantics'
  - restricts side-effects to major time steps
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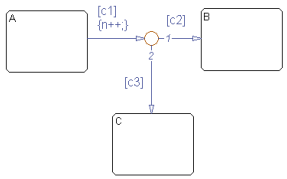

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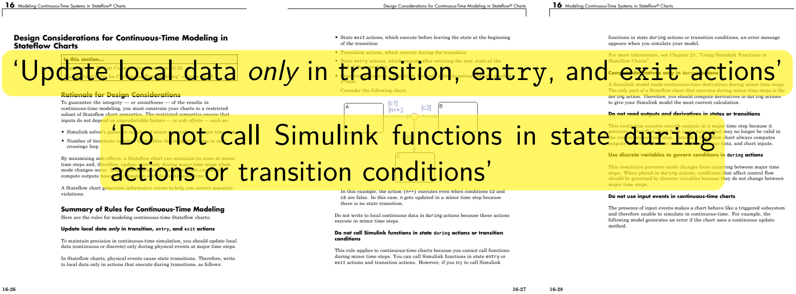
16 Modeling Continuous-Time Systems in Stateflow® Charts

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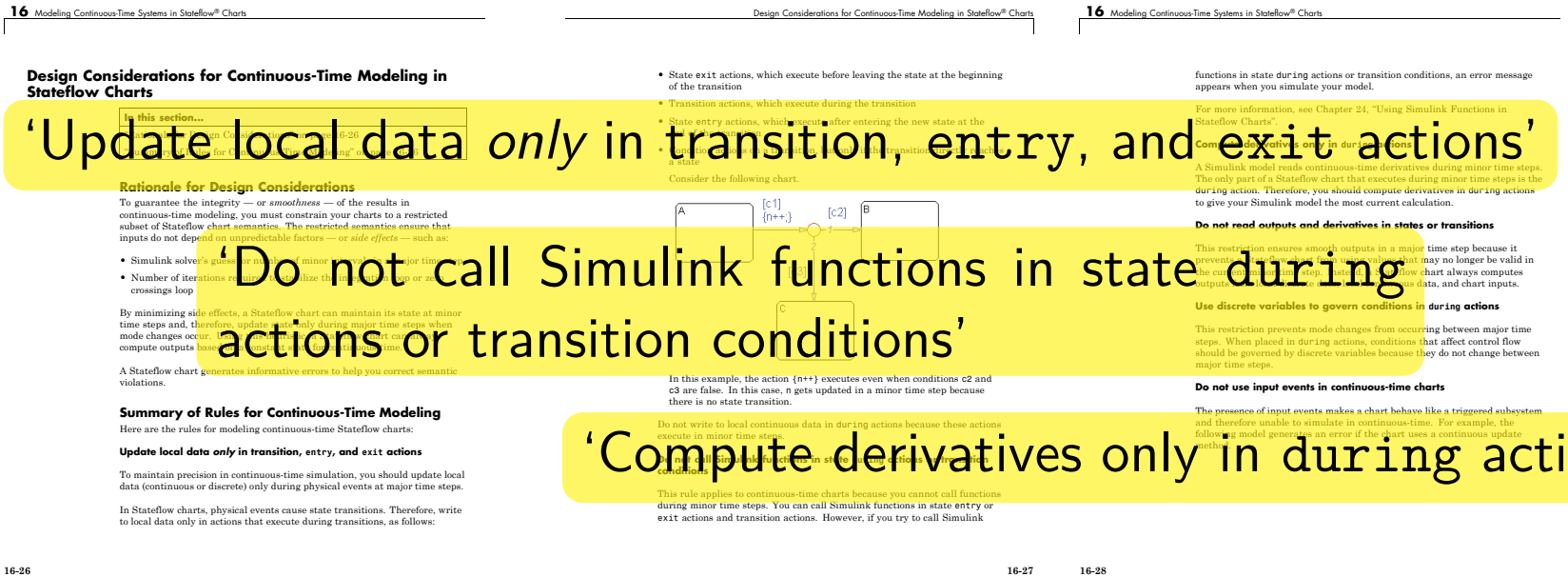

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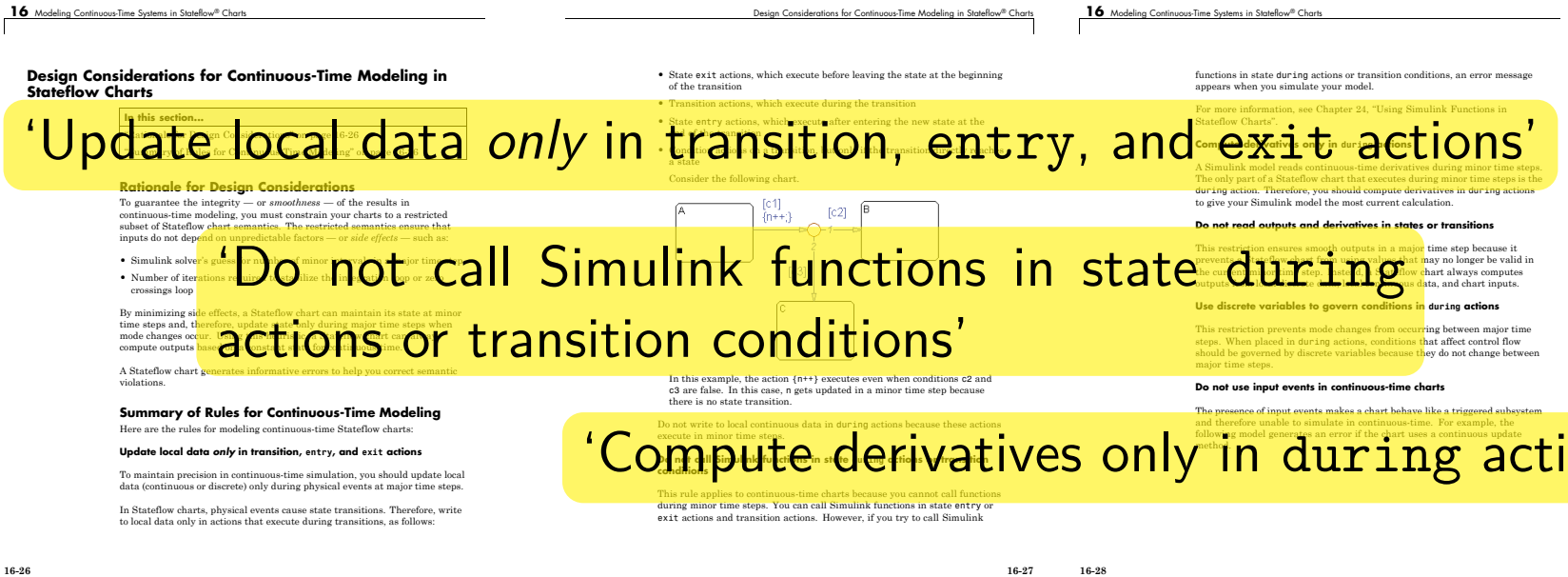

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- 'Restricted subset of Stateflow chart semantics'
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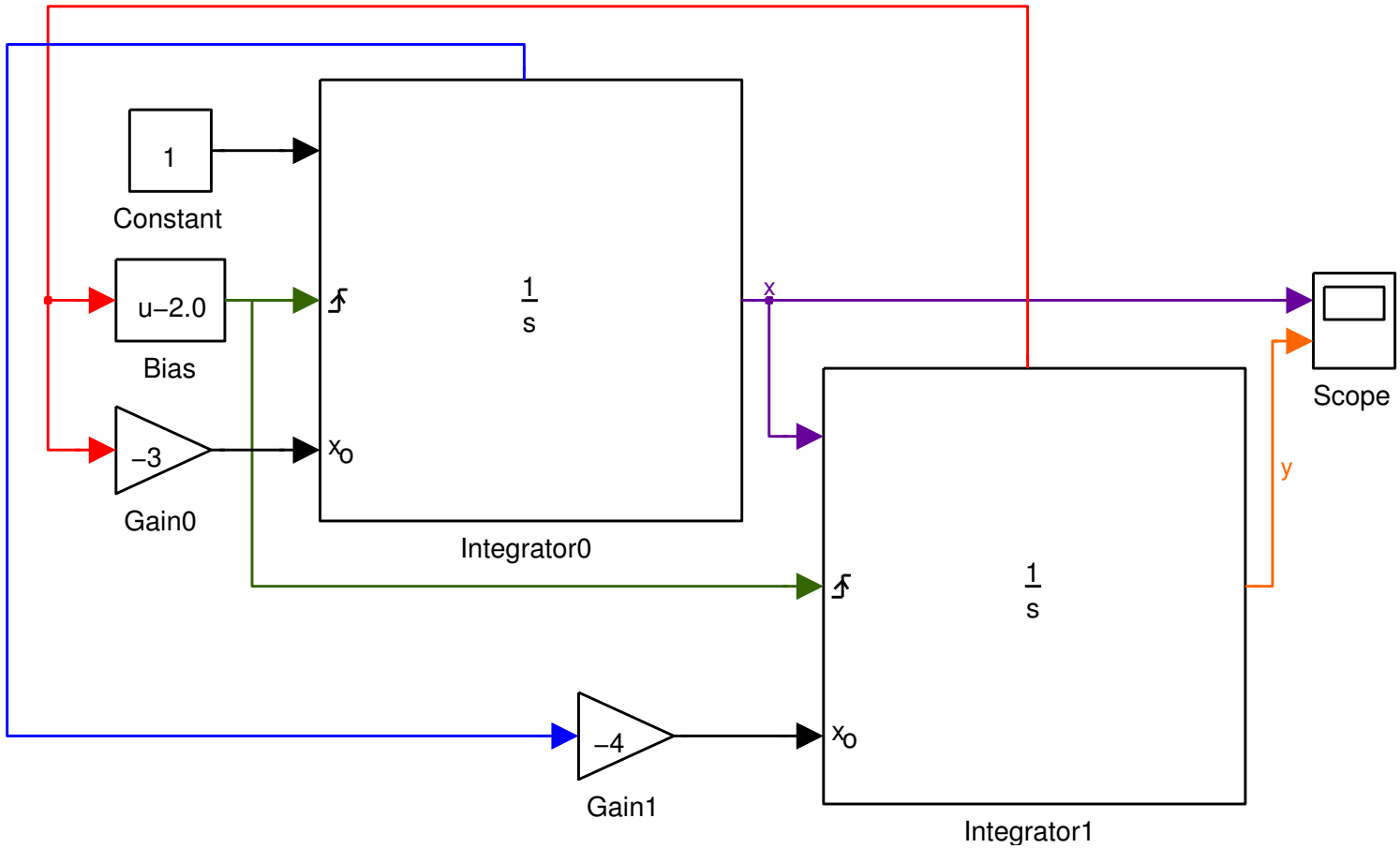
- Such ad-hoc rules can be replaced by a precise type system.

Causality issue: the Simulink state port



The output of the state port is the same as the output of the block's standard output port except for the following case. If the block is reset in the current time step, the output of the state port is the value that would have appeared at the block's standard output if the block had not been reset.
–Simulink Reference (2-685)

Causality issue: the Simulink state port

$$t < 2: \quad x(t) = t,\ y(t) = \frac{t^2}{2}$$

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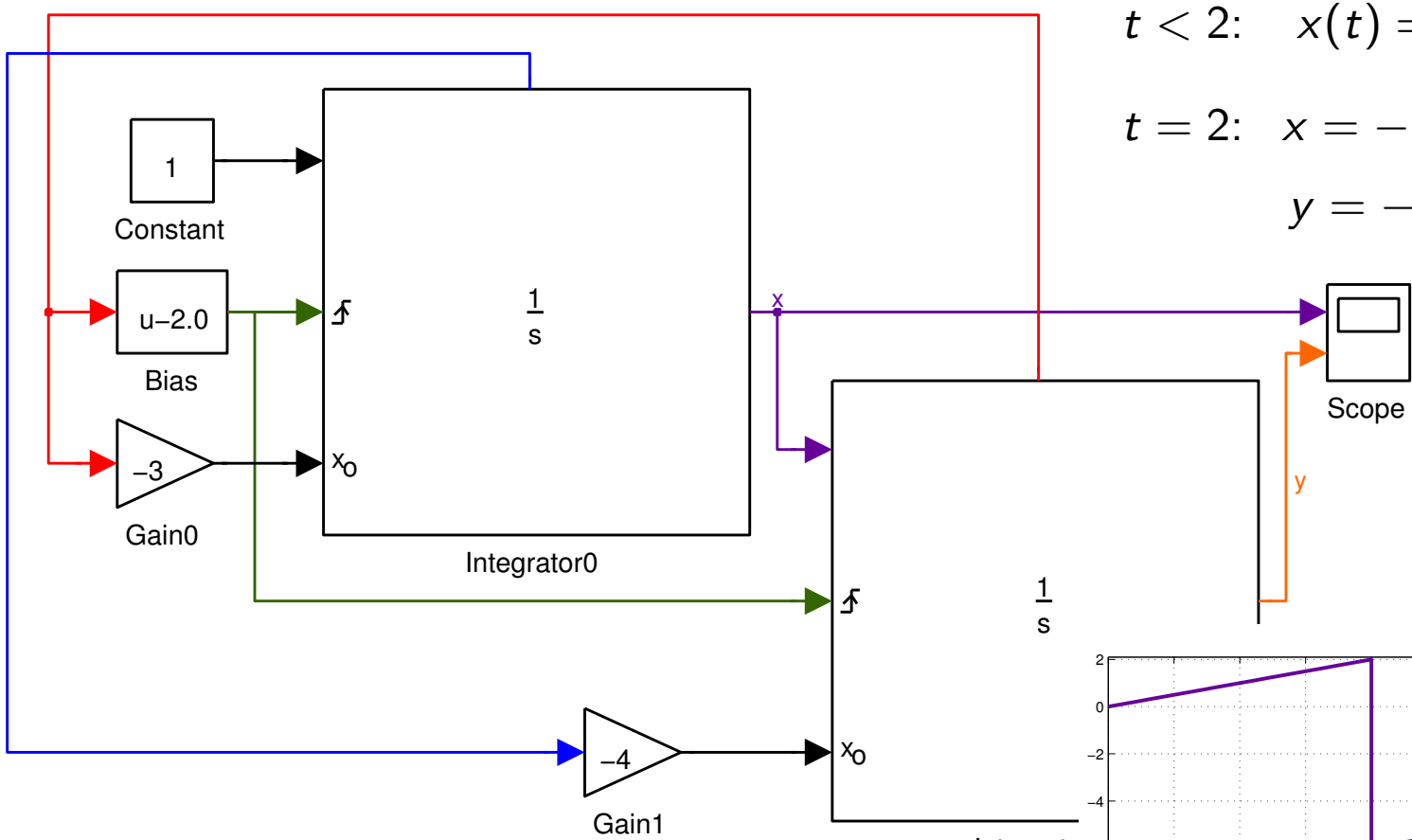

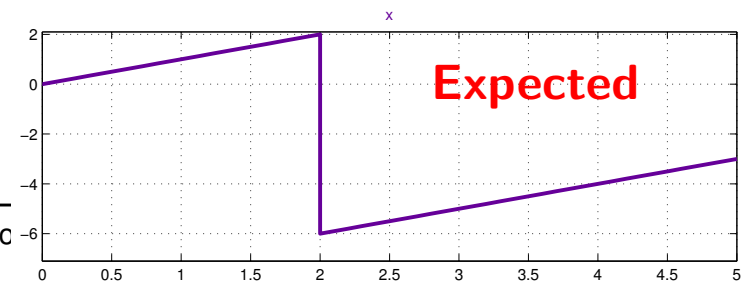

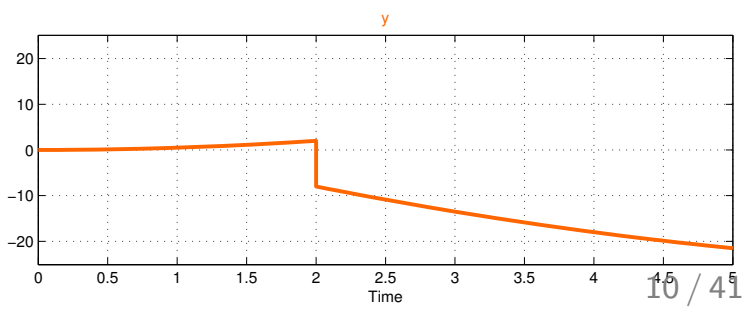

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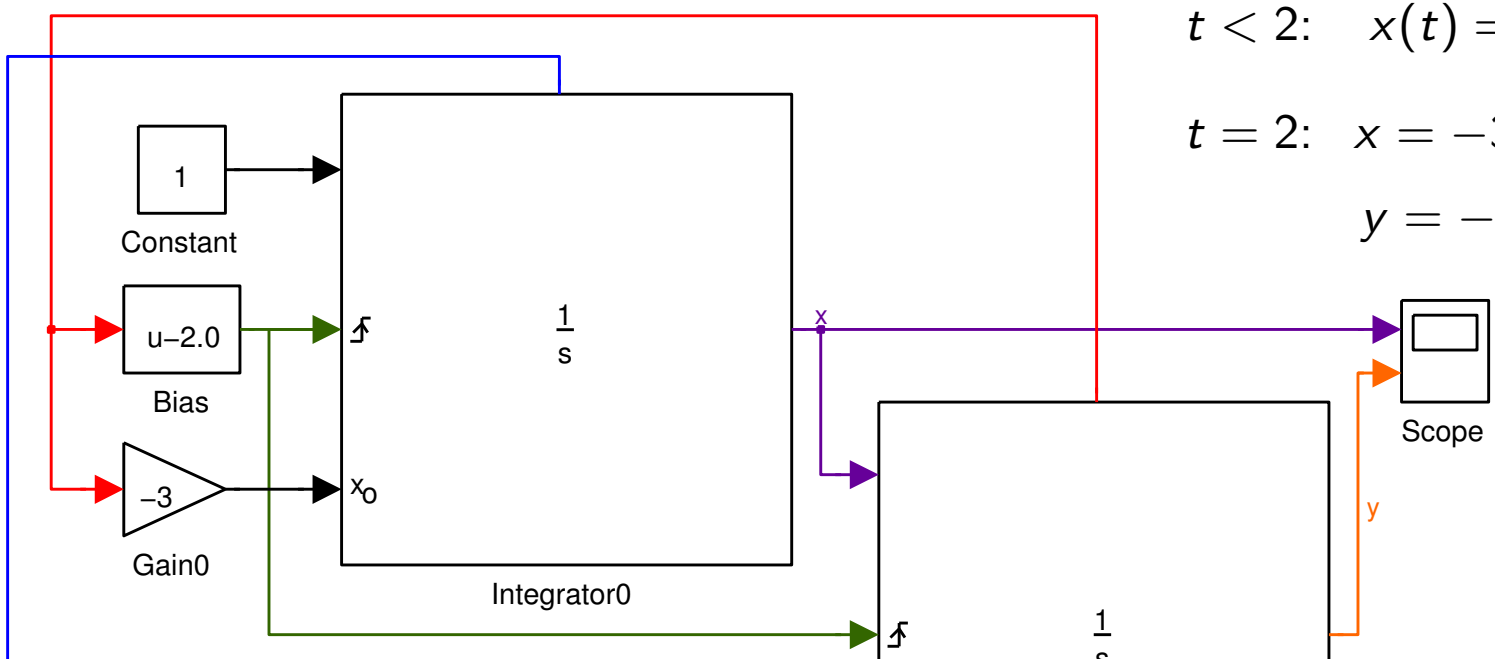
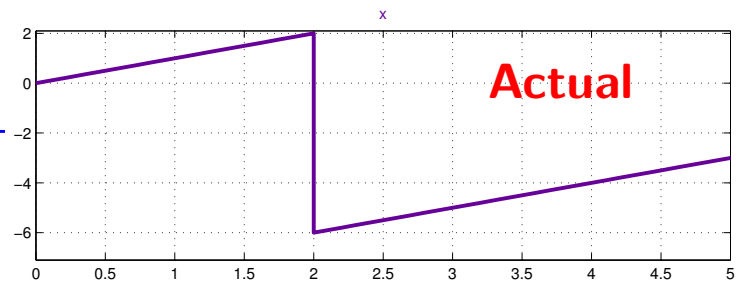
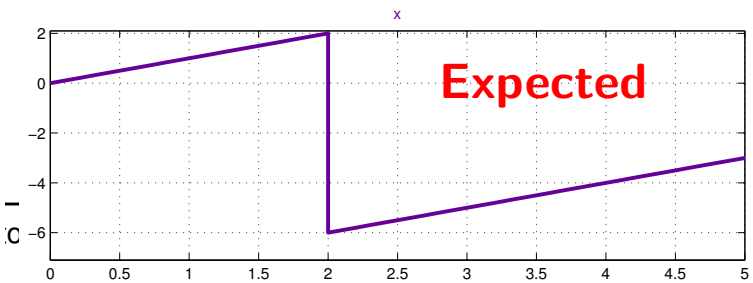
$$t = 2: \quad x = -3 \cdot \text{last}\, y = -6,$$

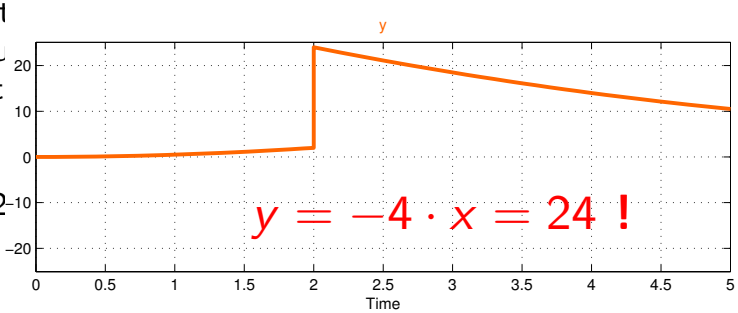
$$y = -4 \cdot \text{last}\, x = -8$$

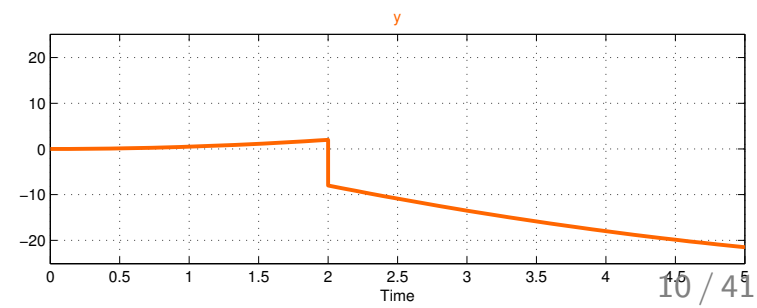






The output of the stat
block's standard outpu
the block is reset in t
state port is the value
standard output if the
–Simulink Reference (2





Excerpt of C code produced by RTW (release R2009)

```
static void mdlOutputs(SimStruct * S, int_T tid)
{ _rtX = (ssGetContStates(S));
  ...
  _rtB = (_ssGetBlockIO(S));
  _rtB->B_0_0_0 = _rtX->Integrator1_CSTATE + _rtP->P_0;
  _rtB->B_0_1_0 = _rtP->P_1 * _rtX->Integrator1_CSTATE;
  if (ssIsMajorTimeStep (S))
    { ...
      if (zcEvent || ...)
        { (ssGetContStates (S))->Integrator0_CSTATE =
            _ssGetBlockIO (S))->B_0_1_0;
        }
      ...
  (_ssGetBlockIO (S))->B_0_2_0 =
    (ssGetContStates (S))->Integrator0_CSTATE;
    _rtB->B_0_3_0 = _rtP->P_2 * _rtX->Integrator0_CSTATE;
    if (ssIsMajorTimeStep (S))
    { ...
      if (zcEvent || ...)
       { (ssGetContStates (S))-> Integrator1_CSTATE =
            (ssGetBlockIO (S))->B_0_3_0;
       }
      ... } ... }
```

Before assignment: integrator state contains 'last' value

$x = -3 \cdot \text{last } y$

After assignment: integrator state contains the new value

$y = -4 \cdot x$

So, y is updated with the new value of x

There is a problem in the treatment of causality.

Current Practice: conclusion

What is the semantics of these tools?

When the manual and implementions diverge, which is right?

There are side effects, global variables, backtracking.

Hard to judge whether the generated code is correct.

What more could we want?

An cleaner integration of discrete and continous time.

Static rejection of bizarre programs.

Real-time (on-line) simulation which is computationally intensive.

Interacting with a numerical solver

It is not always feasible, nor even possible, to calculate the behaviour of a hybrid model analytically.

All major tools thus calculate approximate solutions numerically.

Numerical solvers (e.g., LLNL Sundials CVODE)

Designed by experts.

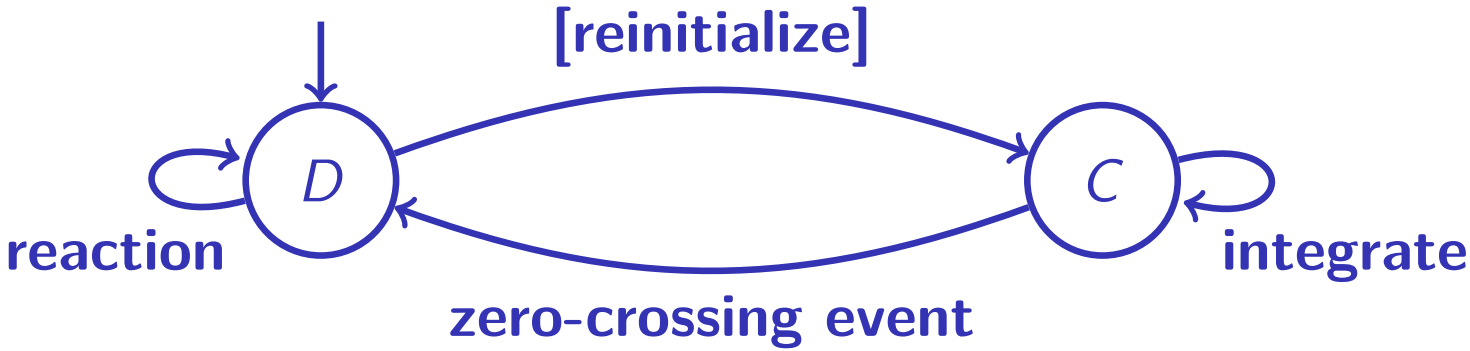
Compute a discrete-time approximations of continuous-time signals.

Subtle: variable step, change order dynamically, explicit/implicit.

Define compilation schemes with solver's internals kept abstract.

The Simulation Engine of Hybrid Systems

Alternate discrete steps and integration steps



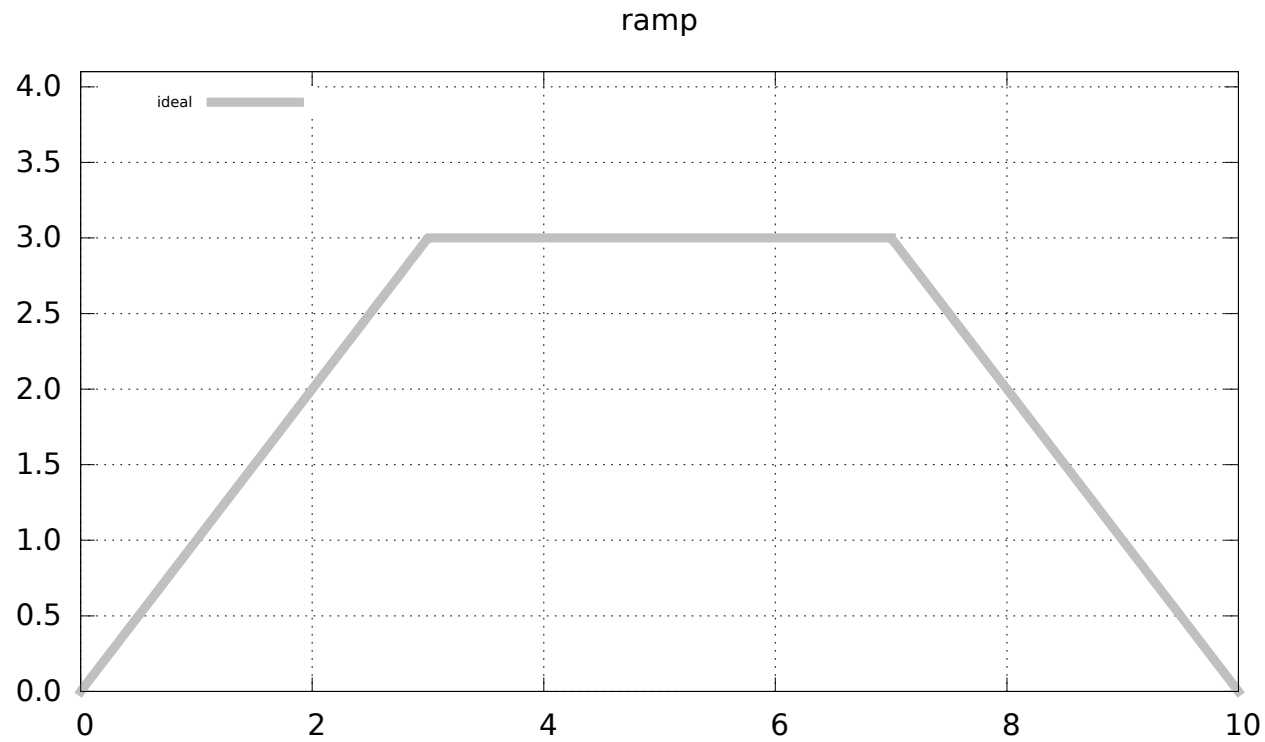
$$\sigma', y' = d_\sigma(t, y) \qquad upz = g_\sigma(t, y) \qquad \dot{y} = f_\sigma(t, y)$$

Properties of the three functions

- d_σ gathers all discrete changes.
- g_σ defines signals for zero-crossing detection.
- f_σ and g_σ should be free of side effects and, better, continuous.

Numerical Integration (Sundials CVODE)

What happens when f_σ has discontinuities? Consider:

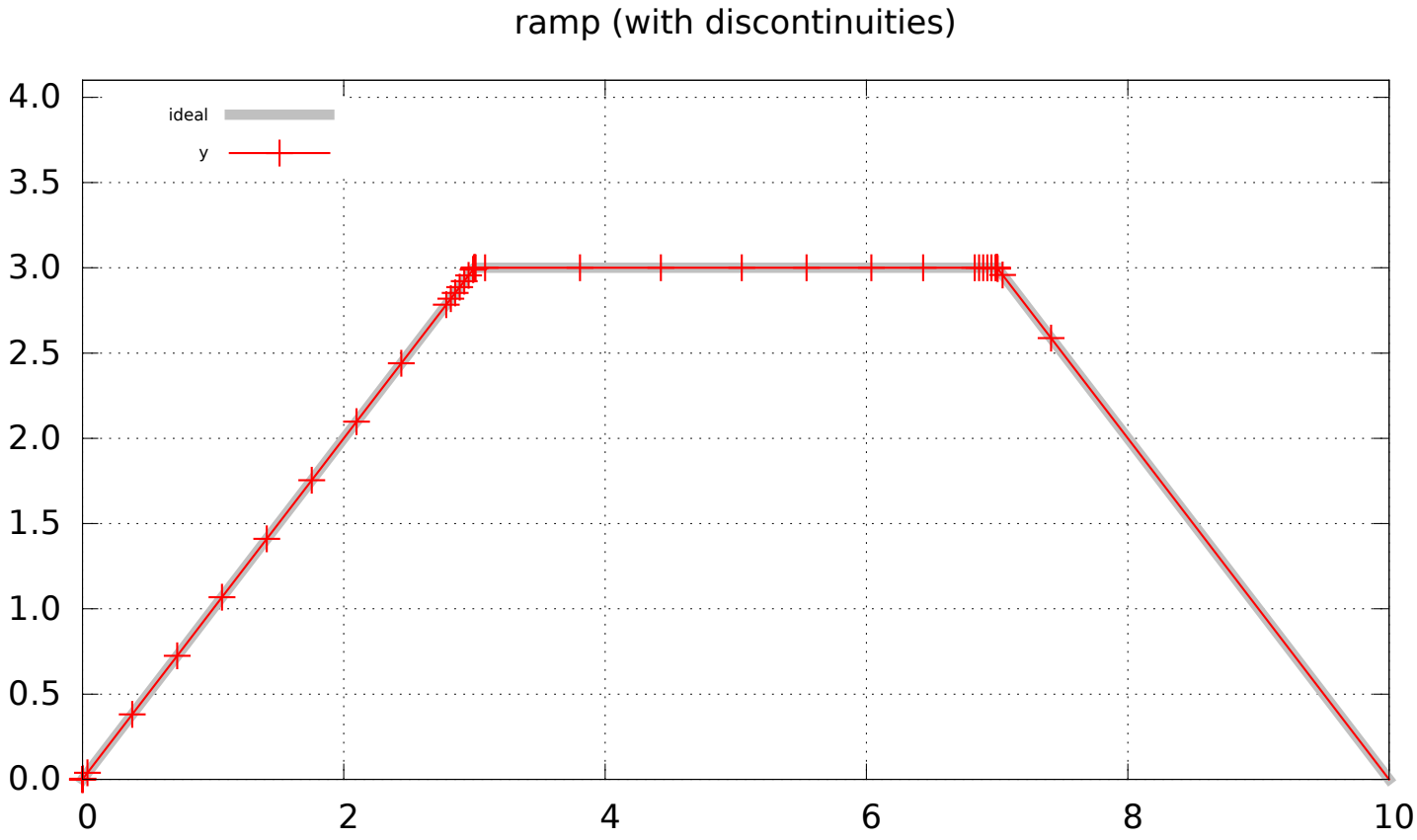


$$\dot{y}(t) = \begin{cases} 1 & \text{if } t < 3 \\ 0 & \text{if } 3 \leq t \leq 7 \\ -1 & \text{if } 7 < t \end{cases}$$

$$y(0) = 0$$

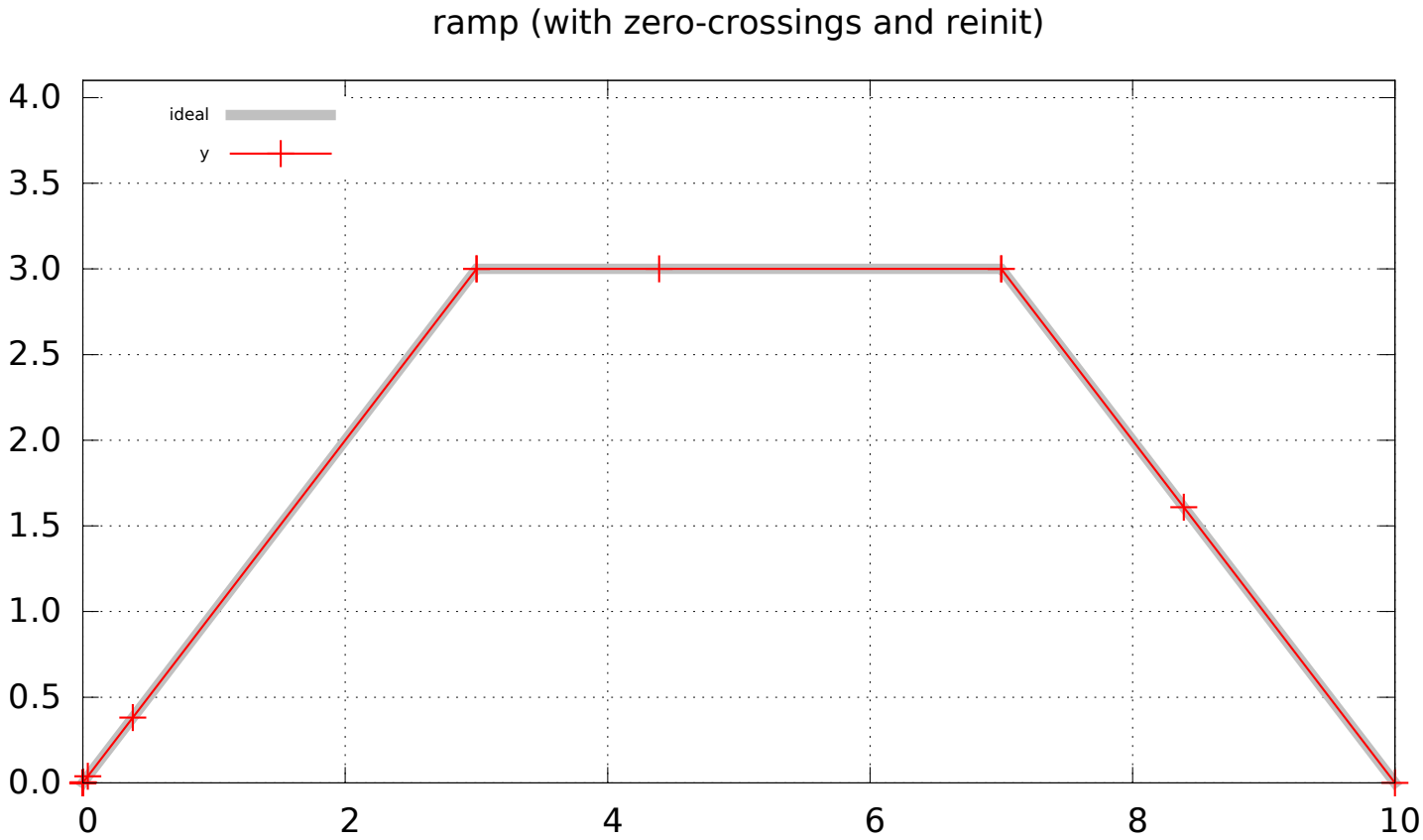
Numerical Integration: a derivative with a discontinuity

The error increases close to the discontinuity, forcing the solver to shrink the step.




```
let f (t, y) =
  if t < 3.0 then 1.0
  else if t <= 7.0 then 0.0
  else -1.0
```

Numerical Integration: discrete state with three modes



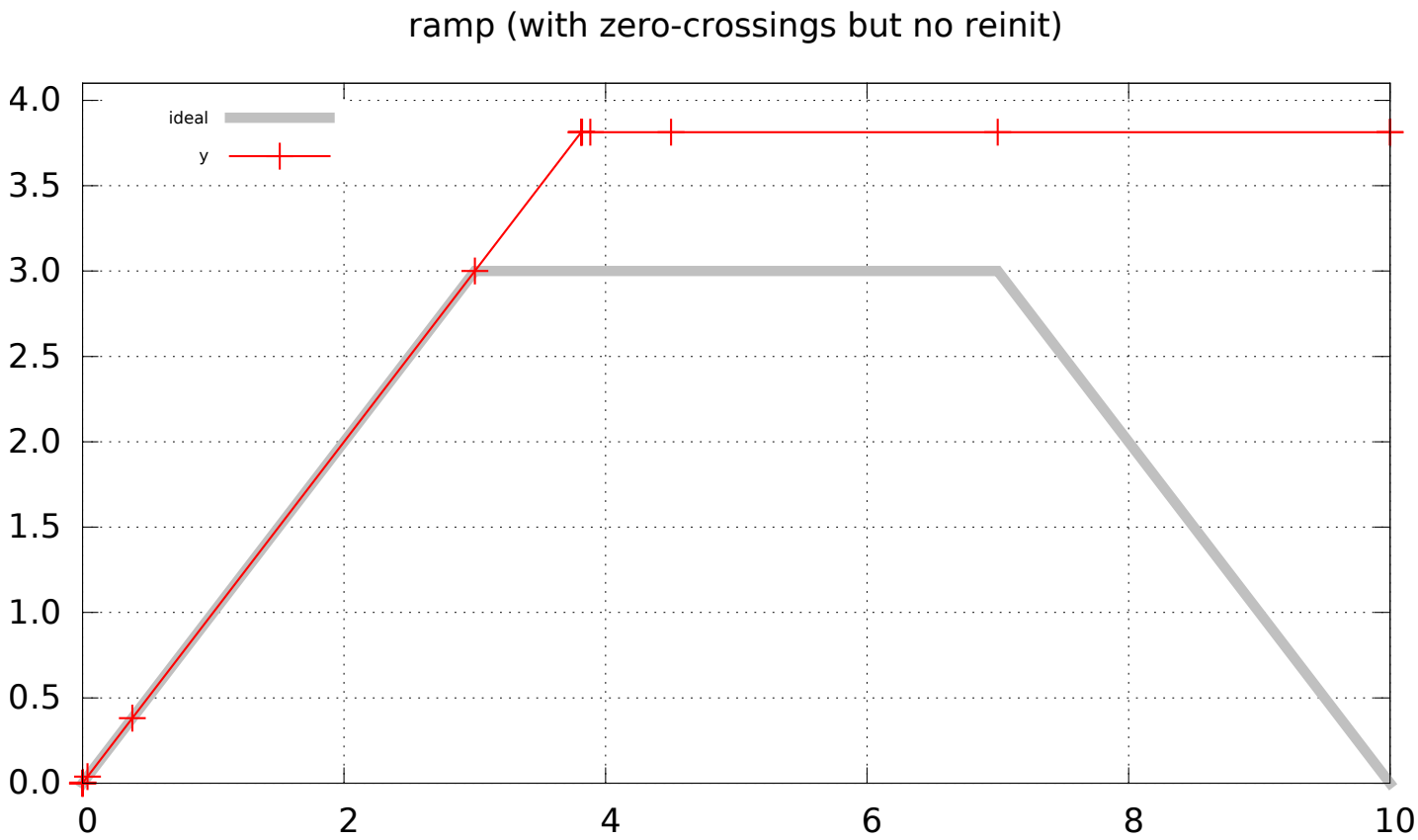

```
let f (discrete_state) (t, y) =
  match discrete_state with
  | RampingUp → 1.0
  | Flat → 0.0
  | RampingDown → −1.0

let g (discrete_state) (t, y) =
  match discrete_state with
  | RampingUp → t −. 3.0
  | Flat → t −. 7.0
  | _ → t
```

```
let discrete_state_i = RampingUp

let d (discrete_state) z (t, y) =
  match discrete_state with
  | RampingUp when z → Flat
  | Flat when z → RampingDown
  | s → s
```

Numerical Integration: with no reinit of the solver

A multi-step solver like Sundials CVODE have an internal state.

It must be reset when signals to be integrated have discontinuities.



Build a Hybrid Modeler on top of a Synchronous Language

Use synchronous constructs for arbitray mix of discrete and continuous.

Divide and Recycle

Recycle existing synchronous languages techniques.

Semantics, static checking, code-generation.

Divide from the code what is for the solver.

Simulate with off-the-shelf numerical solvers.

Be conservative: any synchronous program must be compiled, optimized, and executed as per usual.

These elements are experimented within the language `Zélus`.

Compiler

Zélus is a synchronous language extended with Ordinary Differential Equations (ODEs) to model systems with complex interaction between discrete-time and continuous-time dynamics. It shares the basic principles of Lustre with features from Lucid Synchrone (type inference, hierarchical automata, and signals). The compiler is written

Research

Zélus is used to experiment with new techniques for building hybrid modelers like Simulink/Stateflow and Modelica on top of a synchronous language. The language exploits novel techniques for defining the semantics of hybrid modelers, it provides dedicated type systems to ensure the absence of discontinuities during integration and the

Combinatorial and sequential functions

A signal is a sequence of values. Nothing is said about the actual time to go from one instant to another.

pre is the unit-delay; $\rightarrow$ is the initialization; operations apply pointwise.

```
let add (x,y) = x + y

let node after (n, t) = (c = n) where
  rec c = 0 -> pre(min(tick, n))
  and tick = if t then c + 1 else c
```

When fed into the compiler, we get:

```
val add : int * int -A-> int

val after : int * bool -D-> bool
```

x, y, etc. are infinite sequences of values.

The counter can be instantiated twice in a two state automaton,

```
let node blink (n, m, t) = x where
 automaton
 | On → do x = true   until (after(n, t)) then Off
 | Off → do x = false  until (after(m, t)) then On
```

which returns a value for x that alternates between true for n occurrences of t and false for m occurrences of t.

```
let node blink_reset (r, n, m, t) = x where
 reset
   automaton
   | On  → do x = true  until (after(n, t)) then Off
   | Off → do x = false until (after(m, t)) then On
 every r
```

The type signatures inferred by the compiler are:

```
val blink : int × int × bool -D→ bool

val blink_reset : bool × int × int × bool -D→ bool
```

Generation of Cyclic Sequential Code

A function is translated into a sequential one that computes a single *step*.

- Equations are statically scheduled.
- Straight line code with no recursion nor dynamic allocation.

Example: translation of the function after

```
(* the type structure to store the internal memory *)
type st = { mutable init0: bool; mutable pre0: int }

(* the initialization function *)
let after_reset self = self.init0 ← true

(* the step function *)
let after_step self (n, t) =
  let c = if self.init0 then 0 else self.pre0 in
  let tick = if t then c + 1 else c in
  self.init0 ← false;
  self.pre0 ← min(tick, n);
  (c = n)
```

Hybrid Programs

Up to syntactic details, these are `Scade 6` or `Lucid Synchrone` programs.

A Heat Controller with Hysteresis [2]

```
(* an hysteresis controller for a heater *)
let hybrid heater(active) = temp where
  rec der temp = if active then c −. k ∗. temp else −. k ∗. temp init temp0

let hybrid hysteresis_controller(temp) = active where
  rec automaton
      | Idle →   do active = false until (up(t_min −. temp)) then Active
      | Active → do active = true until (up(temp −. t_max)) then Idle

let hybrid main() = temp where
  rec active = hysteresis_controller(temp)
  and temp = heater(active)
```

[2] This is the hybrid version of one of Nicolas Halbwachs' examples with which he presented Lustre at the Collège de France, in January 2010.

The Bouncing ball [demo]

```
let hybrid bouncing(x0,y0,x'0,y'0) = (x,y) where
 rec
    der x = x' init x0
 and
    der x' = 0.0 init x'0
 and
    der y = y' init y0
 and
    der y' = −. g init y'0 reset up(−. y) → −0.9 ∗. last y'
```

Its type signature is:

```
val bouncing : float × float × float × float -C→ float × float
```

- When $-$y crosses zero, re-initialize the speed y' with $-0.9 *$ last y'.
- When y' is continuous, last y' is the left limit of y'.
- As y' is immediately reset, writing last y' is mandatory —otherwise, y' would instantaneously depend on itself.

Summary of Programming Constructs

- Synchronous constructs: data-flow equations/automata.
- An ODE with initial condition: der x = e init e0
- last x is the left limit of x.
- Detect a zero-crossing (from negative to positive): up(x).
- This defines a discrete instant, that is, an event.
- All discrete changes must occur on an event. E.g.,:

```
let hybrid f(x, y) = (v, z1, z2) where
  rec v = present z1 → 1 | z2 → 2 init 0
  and z1 = up(x)
  and z2 = up(y)
```

```
val f : float × float -C→ int × zero × zero
```

- If x = up(e), all handlers using x are governed by the same event.

Three difficulties

Semantics

- An ideal semantics to say which program make sense;
- useful to prove that compilation is correct.

Ensure that continuous and discrete time signals interfere correctly.

- Discrete time should stay logical and independent on when the solver decides to stop.
- Otherwise, we get the bizarre behaviors seen previously.

Ensure that fix-points exist and code can be scheduled.

- Algebraic loops must be statically detected.
- Restrict the use of last x so that signals are proved to be continuous during integration.

A Non-standard Semantics for Hybrid Modelers [JCSS'12]

We proposed to build the semantics on non-standard analysis.

der y = z init 4.0 and z = 10.0 −. 0.1 ∗. y and k = y +. 1.0

defines signals y, z and k, where for all $t \in \mathbb{R}^+$:

$$\frac{dy}{dt}(t) = z(t) \quad y(0) = 4.0 \quad z(t) = 10.0 - 0.1 \cdot y(t) \quad k(t) = y(t) + 1$$

What would be the value of y if it were computed by an ideal solver taking an infinitesimal step of duration ∂?

${}^\star y(n)$ stands for the values of y at instant $n\partial$, with $n \in {}^\star\mathbb{N}$ a non-standard integer.

$$\begin{aligned} {}^\star y(0) &= 4 & {}^\star z(n) &= 10 - 0.1 \cdot {}^\star y(n) \\ {}^\star y(n+1) &= {}^\star y(n) + {}^\star z(n) \cdot \partial & {}^\star k(n) &= {}^\star y(n) + 1 \end{aligned}$$

Non standard semantics [JCSS'12]

Let ${}^\star\mathbb{R}$ and ${}^\star\mathbb{N}$ be the non-standard extensions of $\mathbb{R}$ and $\mathbb{N}$.

Let $\partial \in {}^\star\mathbb{R}$ be an infinitesimal, i.e., $\partial > 0, \partial \approx 0$.

Let the global time base or base clock be the infinite set of instants:

$$\mathbb{T}_\partial = \{t_n = n\partial \mid n \in {}^\star\mathbb{N}\}$$

$\mathbb{T}_\partial$ inherits its total order from ${}^\star\mathbb{N}$. A sub-clock $T \subset \mathbb{T}_\partial$.

What is a discrete clock?

> *A clock T is termed discrete if it is the result of a zero-crossing or a sub-sampling of a discrete clock. Otherwise, it is termed continuous.*

If $T \subseteq \mathbb{T}$, we write ${}^\bullet T(t)$ for the immediate predecessor of t in T and $T^\bullet(t)$ for the immediate successor of t in T.

A signal is a partial function from $\mathbb{T}$ to a set of values.

Semantics of basic operations

Replay the classical semantics of a synchronous language.

An ODE with reset on clock T: $\text{der } x = e \text{ init } e_0 \text{ reset } z \to e_1$

$$^\star x(t_0) = {}^\star e_0(0) \text{ if } t_0 = \min T$$
$$^\star x(t) = \text{if } {}^\star z(t) \text{ then } {}^\star e_1(t) \text{ else } {}^\star x(^\bullet T(t)) + \partial \cdot {}^\star e(^\bullet T(t)) \text{ if } t \in T$$

last x if x is defined on clock T

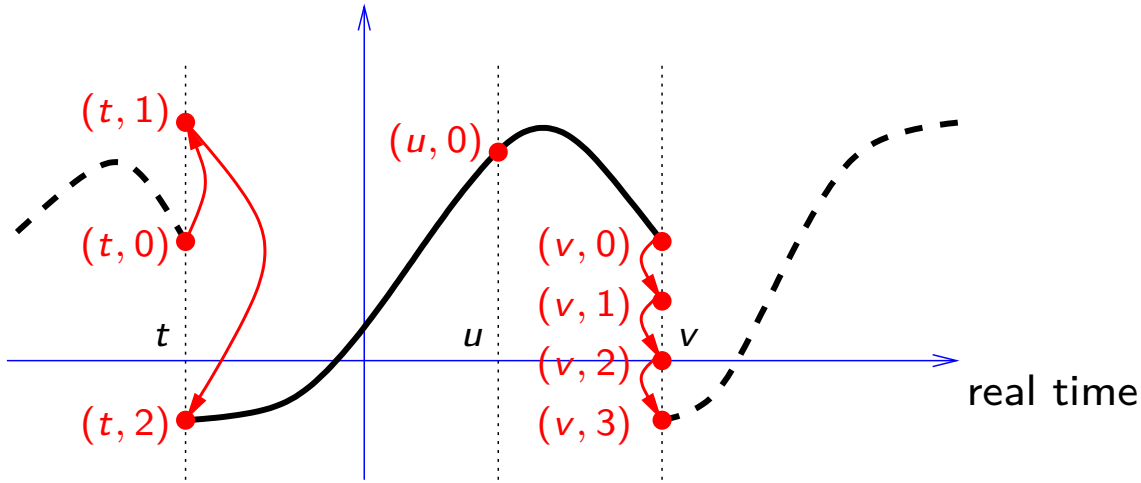
$$^\star \text{last } x(t) = {}^\star x(^\bullet T(t))$$

Zero-crossing up(x) on clock T

$$^\star \text{up}(x)(t_0) = \text{false if } t_0 = \min T$$
$$^\star \text{up}(x)(t) = ({}^\star x(^\bullet T(t) < 0) \wedge ({}^\star x(t) \geq 0) \text{ if } t \in T$$

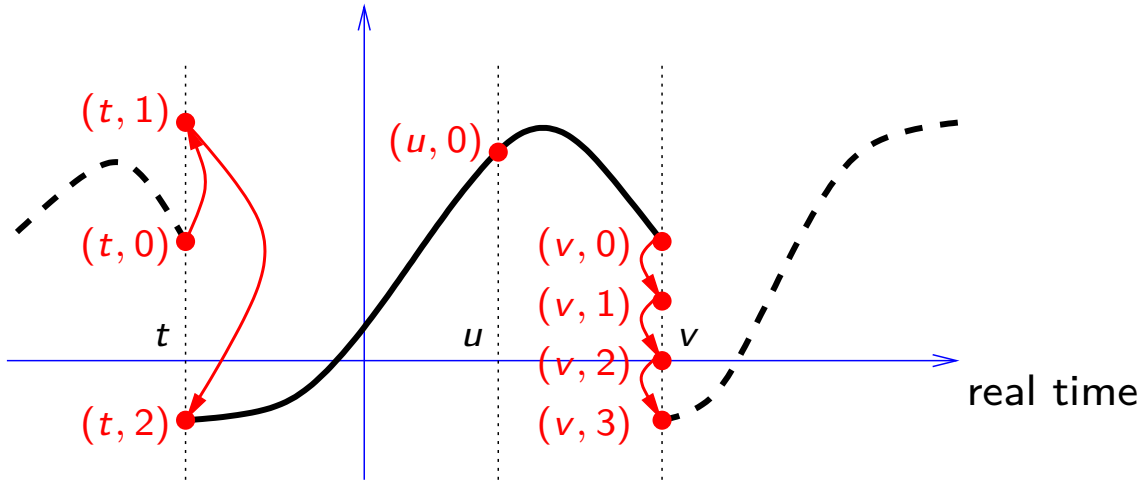
Non-standard time vs. Super-dense time

- Maler et al., Edward Lee et al. super-dense time modeling $\mathbb{R} \times \mathbb{N}$

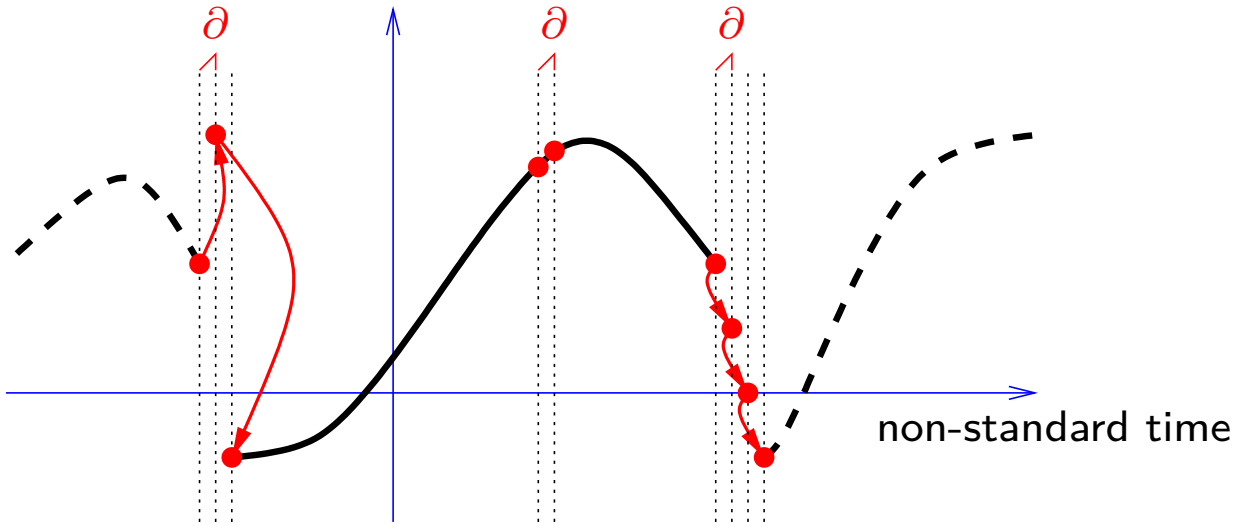


Non-standard time vs. Super-dense time

- Maler et al., Edward Lee et al. super-dense time modeling $\mathbb{R} \times \mathbb{N}$



- non-standard time modeling $\mathbb{T}_\partial = \{n\partial \mid n \in {}^\star\mathbb{N}\}$



Typing: mixing discrete (logical) time and continuous time

The following two parallel composition make sense.

Discrete time: the clock should be discrete

```
let node sum(x) = cpt where
  rec cpt = 0 → pcpt
  and pcpt = pre(cpt) + x
```

Continuous time: the clock should be continuous

```
let hybrid bouncing(y0, y'0) = o where
  rec der y = y' init y0
  and der y' = −.g init y'0
  and o = y +. 10.0
```

The following do not make sense

At what clock should we compute cpt?

```
rec der t = 1.0 init 0.0
and cpt = 0.0 → pre(cpt) + t
```

Intuition

Distinguish functions with three kinds A/D/C.

- Combinatorial function get kind A (for "any").
- Discrete-time (synchronous) functions get kind D (for "discrete").
- Continuous-time (hybrid) functions get kind C (for "continuous").

Explicitly relate simulation and logical time

All discontinuities and side effects must be aligned with a zero-crossing instant.

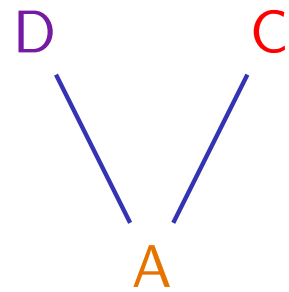
```
let node sum(x) = cpt where
  rec cpt = 0.0 → pcpt
  and pcpt = pre(cpt) +. x

let hybrid correct (z) = (time, y) where
  rec der time = 1.0 init 0.0
  and y = present up(z) → sum(time) init 0.0
```

Basic typing [LCTES'11,EMSOFT'12]

A simple ML type system with effects.

The type language

$$
\begin{array}{lcl}
bt & ::= & \mathsf{float} \mid \mathsf{int} \mid \mathsf{bool} \mid \mathsf{zero} \\
t & ::= & bt \mid t \times t \mid \beta \\
\sigma & ::= & \forall \beta_1, ..., \beta_n . t \xrightarrow{k} t \\
k & ::= & \mathsf{D} \mid \mathsf{C} \mid \mathsf{A}
\end{array}
$$



Initial conditions

$$
\begin{array}{lcl}
(+) & : & \mathsf{int} \times \mathsf{int} \xrightarrow{\mathsf{A}} \mathsf{int} \\
\texttt{if} & : & \forall \beta . \mathsf{bool} \times \beta \times \beta \xrightarrow{\mathsf{A}} \beta \\
(=) & : & \forall \beta . \beta \times \beta \xrightarrow{\mathsf{D}} \mathsf{bool} \\
\mathsf{pre}(\cdot) & : & \forall \beta . \beta \xrightarrow{\mathsf{D}} \beta \\
\mathsf{up}(\cdot) & : & \mathsf{float} \xrightarrow{\mathsf{C}} \mathsf{zero}
\end{array}
$$

Causality loops

Some programs are well typed but have algebraic loops.

Which programs should we accept?

- OK to reject (no solution).

  rec x = x + 1

- OK as an algebraic constraint (e.g., Simulink and Modelica).

  rec x = 1 − x

  But NOK for sequential code generation.

- last x does not necessarily break causality loops!

  rec x = last x + 1

  But this program is OK:

  rec der x = 1.0 init 0.0 reset z → t
  and y = x +. 1.0
  and t = last y

Can we find a simple and uniform justification?

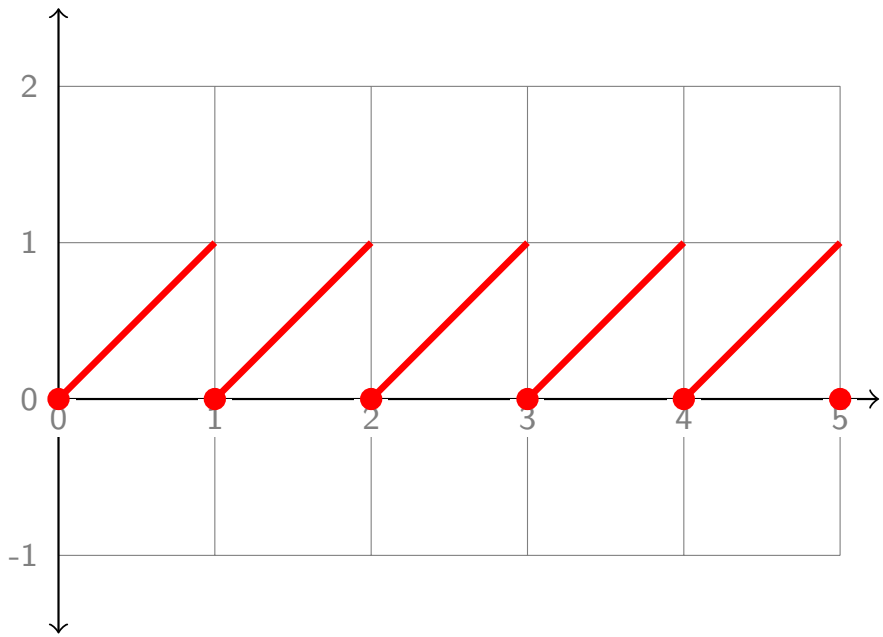
ODEs with reset

Consider the sawtooth signal $y : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that:

$$\frac{dy}{dt}(t) = 1 \quad y(t) = 0 \text{ if } t \in \mathbb{N}$$

written:

$$\text{der y} = 1.0 \text{ init } 0.0 \text{ reset up(y} -. 1.0) \rightarrow 0.0$$



ODEs with reset

Consider the sawtooth signal $y : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that:

$$\frac{dy}{dt}(t) = 1 \quad y(t) = 0 \text{ if } t \in \mathbb{N}$$

written:

der y = 1.0 init 0.0 reset up(y −. 1.0) → 0.0

The ideal non-standard semantics is:

$$\begin{aligned} {}^\star y(0) &= 0 & {}^\star y(n) &= \text{if } {}^\star z(n) \text{ then } 0.0 \text{ else } {}^\star ly(n) \\ {}^\star ly(n) &= {}^\star y(n-1) + \partial & {}^\star c(n) &= ({}^\star y(n) - 1) \geq 0 \\ {}^\star z(0) &= \text{false} & {}^\star z(n) &= {}^\star c(n) \wedge \neg {}^\star c(n-1) \end{aligned}$$

This set of equation is not causal: ${}^\star y(n)$ depends on itself.

Accessing the left limit of a signal

There are two ways to break this cycle:

- consider that the effect of a zero-crossing is delayed by one cycle, that is, the test is made on ${}^\star z(n-1)$ instead of on $z(n)$, or,

- distinguish the current value of ${}^\star y(n)$ from the value it would have had were there no reset, namely ${}^\star ly(n)$.

Testing a zero-crossing of ly (instead of y),

$$ {}^\star c(n) = ({}^\star ly(n) - 1) \geq 0, $$

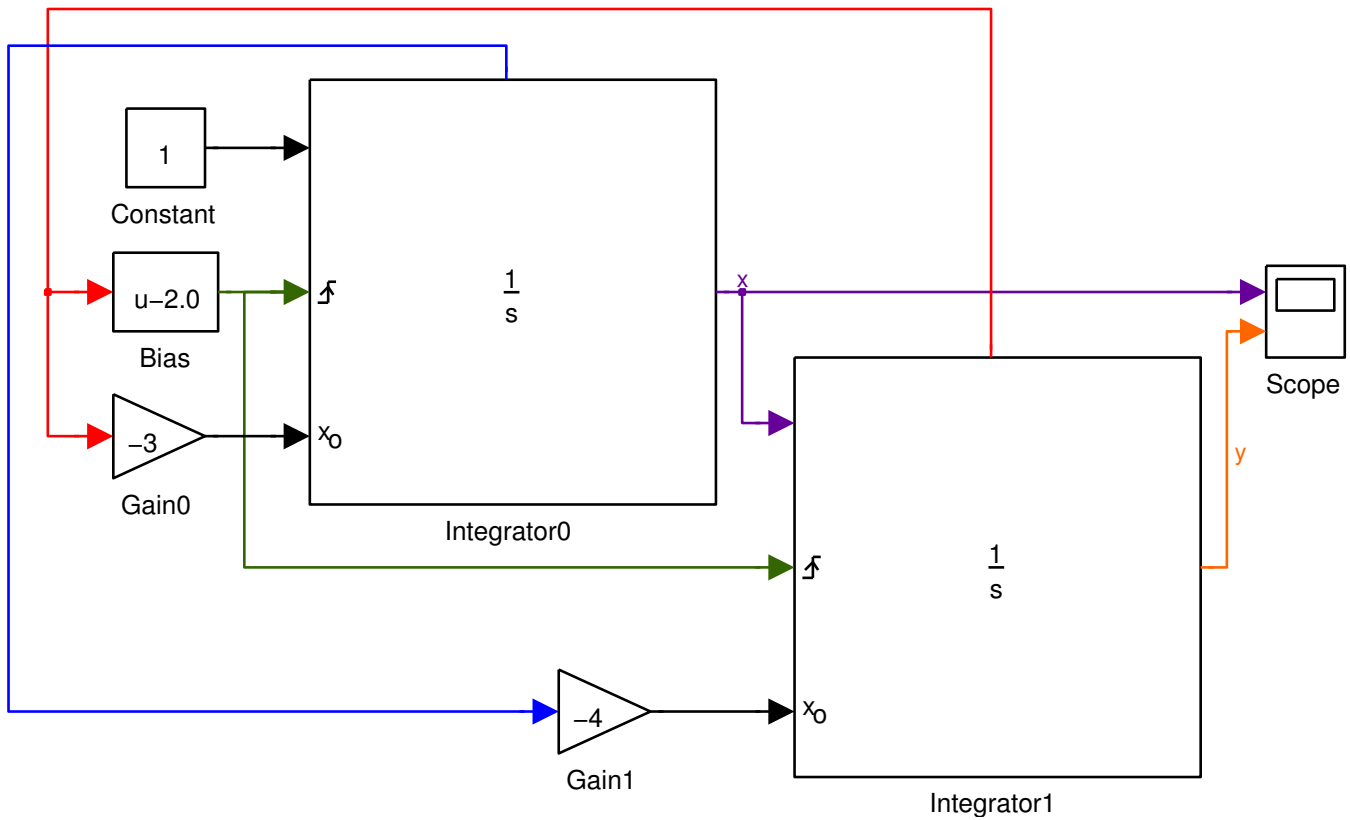
gives a program that is causal since ${}^\star y(n)$ no longer depends instantaneously on itself.

der y = 1.0 init 0.0 reset up(last y −. 1.0) → 0.0

An explanation of the causality 'issue' of Simulink

The source program

rec der x = 1.0 init 0.0 reset z → −3.0 ∗. last y
and der y = x init 0.0 reset z → −4.0 ∗. last x
and z = up(last x −. 2.0)



An explanation of the causality 'issue' of Simulink

The source program

rec der x $= 1.0$ init 0.0 reset z $\rightarrow -3.0 *.$ last y
and der y $=$ x init 0.0 reset z $\rightarrow -4.0 *.$ last x
and z $=$ up(last x $-.$ 2.0)

Its non-standard interpretation

$$
\begin{array}{rcl}
{}^\star x(n) & = & \text{if } {}^\star z(n) \text{ then } -3 \cdot {}^\star y(n-1) \text{ else } {}^\star x(n-1) + \partial \\
{}^\star y(n) & = & \text{if } {}^\star z(n) \text{ then } -4 \cdot {}^\star x(n-1) \text{ else } {}^\star y(n-1) + \partial \cdot {}^\star x(n-1) \\
& \vdots &
\end{array}
$$

Explanation

- The first two equations are scheduled this way so ${}^\star x(n-1)$ is lost.
- This is a scheduling bug: the sequential code lacks a copy variable.

Causality Analysis [HSCC'14]

Every feedback loop must cross a delay.

Intuition: associate a 'time stamp' to every expression and require that the relation $<$ between time stamps is a strict partial order.

The type language

$$\begin{array}{lcl} \sigma & ::= & \forall \alpha_1, ..., \alpha_n : C.\ ct \xrightarrow{k} ct \\ ct & ::= & ct \times ct \mid \alpha \\ k & ::= & \mathsf{D} \mid \mathsf{C} \mid \mathsf{A} \end{array}$$

Precedence relation:

$$C ::= \{\alpha_1 < \alpha_1', ..., \alpha_n < \alpha_n'\}$$

$C \vdash ct_1 < ct_2$ means that ct_1 precedes ct_2 according to C.

Associate a type that express input/output dependences. E.g.,

let node plus(x, y) = x + 0 → pre y

We get: $f : \forall \alpha_1, \alpha_2.\ \alpha_1 \times \alpha_2 \xrightarrow{D} \alpha_1$

- der x breaks a loop: der temp = c −. temp init 20.0 is correct.
- last(x) breaks a loop in a discrete context.

The following is rejected; the next is accepted.

```
rec der y' = −. g init 0.0 reset up(−.y) → −0.9 ∗. y'
and der y = y' init y0
```

```
rec der y' = −. g init 0.0 reset up(−.y) → −0.9 ∗. last y'
and der y = y' init y0
```

Theorem: [HSCC 14] Well typed programs define continuous signals during integration.

The proof is based on the non-standard synchronous semantics.

Comparison with existing tools

Simulink/Stateflow (Mathworks)

- Integrated treatment of automata *vs* two distinct languages
- More rigid separation of discrete and continuous behaviors

Modelica

- Do not handle DAEs
- Our proposal for automata has been integrated into version 3.3

Ptolemy (E.A. Lee et al., Berkeley)

- A unique computational model: synchronous
- Everything is compiled to sequential code (not interpreted)

Synchronous languages should and can properly treat hybrid systems